Transient Analysis of First-Order Circuits: Approaches and Recommendations

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Abstract

First-order systems occur frequently in nature. A first-order system can be defined as any system that can absorb energy through a storage element and release that stored energy. In electric circuits, there are two circuit elements that have the capability to store energy. A capacitor stores energy in the electric field within its dielectric medium, and an inductor stores energy in the magnetic field induced by the current flowing through its conductors. Hence, for electric circuits, any circuit that contains a single capacitor or a single inductor in addition to resistors, voltage and/or current sources can be classified as a first-order circuit. First-order circuits are called RC or RL circuits, respectively, and can be described by a first-order differential equation.

The analysis of first-order circuits involves examining the behavior of the circuit as a function of time before and after a sudden change in the circuit due to switching actions. There are several approaches used to analyze first-order circuits. The most popular two are the differential equation approach and the general step-by-step approach.

This paper presents both approaches for performing a transient analysis in first-order circuits: the differential equation approach where a differential equation is written and solved for a given circuit, and the step-by-step approach where the advantage of a priori knowledge of the form of the solution is taken into account. A solved example using both approaches is provided. The performance and attitude of students with respect to each approach in the Electric Circuits course at Ohio Northern University are assessed and the result of this assessment is presented.

1. Introduction

Electric Circuits Analysis is a required course in many engineering programs. At Ohio Northern University, the course is required from students majoring in electrical, computer, and mechanical engineering and recently from students in the newly developed engineering education program (Bachelor of Science in Engineering Education). It is also an elective course for students in civil engineering. The course is 4 semester credit hours, which consists of three 50- minute lectures and a 2-hour associated laboratory each week. The content of the course includes: resistive circuits, analysis techniques, circuits with operational amplifiers, first-order circuit transient analysis, ac steady-state analysis, ideal transformers, steady-state power analysis, and balanced polyphase circuits.

The emphasis of this paper is on the topic of first-order circuit transient analysis. Generally, first-order circuits have only one storage element; i.e., one capacitor or one inductor present in the circuit. Although there are special conditions in which more than one capacitor or inductor can be present and still be considered a first-order circuit – i.e., whenever the additional storage elements are not linearly independent storage elements – the paper simplifies the presentation by focusing on the case of a single capacitor or single inductor present in the first-order circuit.

The issue with transient analysis of first-order circuits is the fact that a majority of students find this topic to be challenging[1, 2]. In resistive circuits, all the equations are algebraic because the interconnected devices satisfy Ohm's law, Kirchhoff's voltage and current laws. However, capacitors and inductors have differential or integral voltage-current relationships. Hence, the interconnected devices lead to electric circuits that must satisfy both algebraic equations as well as integral or differential relationships.

The first-order transient circuits can be solved through several methods; the most popular methods are the differential equation approach and the general step-by-step approach[2, 3, 4]. This paper covers both approaches and is organized as follows: Section 2 covers some mathematical preliminaries, Section 3 explains both approaches along with a solved example using both methods. A comparison of students' performance using both approaches is covered in section 4. Section 5 concludes the paper with a recommendation.

2. First-Order Circuit Preliminaries

Electric circuits that contain only resistors in addition to current and/or voltage sources are called static circuits and are represented by algebraic equations. The circuits that contain storage elements (capacitors and inductors) in addition to current and/or voltage sources are called dynamic circuits and are represented by differential equations. Capacitors and inductors are called storage elements due to their ability to store energy. Inductors have the ability of storing magnetic energy in their field, whereas capacitors have the ability of storing electric energy in their field.

The current-voltage relationship for a capacitor is given as

$$i(t) = C\frac{dv(t)}{dt} \tag{1}$$

where $v(t)$ is the voltage across the capacitor in volts, and $i(t)$ is the current through the capacitor in amperes. We can rewrite (1) as

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(t)dt + v(t_0) \tag{2}$$

From (1) and (2), we can conclude that the current through a capacitor is zero for a constant voltage across it, which leads to the fact that the capacitor acts as an open circuit for this case, and the voltage cannot change abruptly.

The voltage-current relationship for an inductor is given as

$$v(t) = L\frac{di(t)}{dt} \tag{3}$$

where $v(t)$ is the voltage across the inductor in volts, and $i(t)$ is the current through the inductor in amperes. We can rewrite (3) as

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v(t)dt + i(t_0) \tag{4}$$

From (3) and (4), we can conclude that the induced voltage across an inductor is zero for a constant current which leads to the fact that the inductor acts as a short circuit for this case, and the current cannot change abruptly.

Based on the above, any circuit that contains one storage element can be represented by a first-order differential equation. Hence, these circuits are called first-order circuits.

3. Analysis Techniques

The analysis of first-order circuits requires the solution of differential equations. The complete solution consists of two parts: the homogeneous solution and the particular solution. The particular solution of a first-order circuit with DC sources and switching action is the steady-state response and also called the forced response. The homogenous solution consists of the characteristic mode of the first-order circuit, which decays to zero after a few time constants, and is also called the transient response.

There are two popular techniques in solving first-order RC and RL circuits:

- Differential Equation Approach
  There are five major steps in finding the complete response of a given first order-circuit:
  1. Determine initial conditions on the capacitor voltage and/or inductor current.
  2. Find the differential equation for either capacitor voltage or inductor current (mesh/loop/nodal analysis).
  3. Determine the form of the homogeneous solution (i.e., the characteristic mode).
  4. Determine the particular solution.
  5. Apply the initial condition to the complete solution to determine the unknown coefficient in the homogeneous solution.

- Step-by-Step Approach
  There are 7 major steps in finding the complete response of a given first order circuit, these steps are presented in Figure 1.

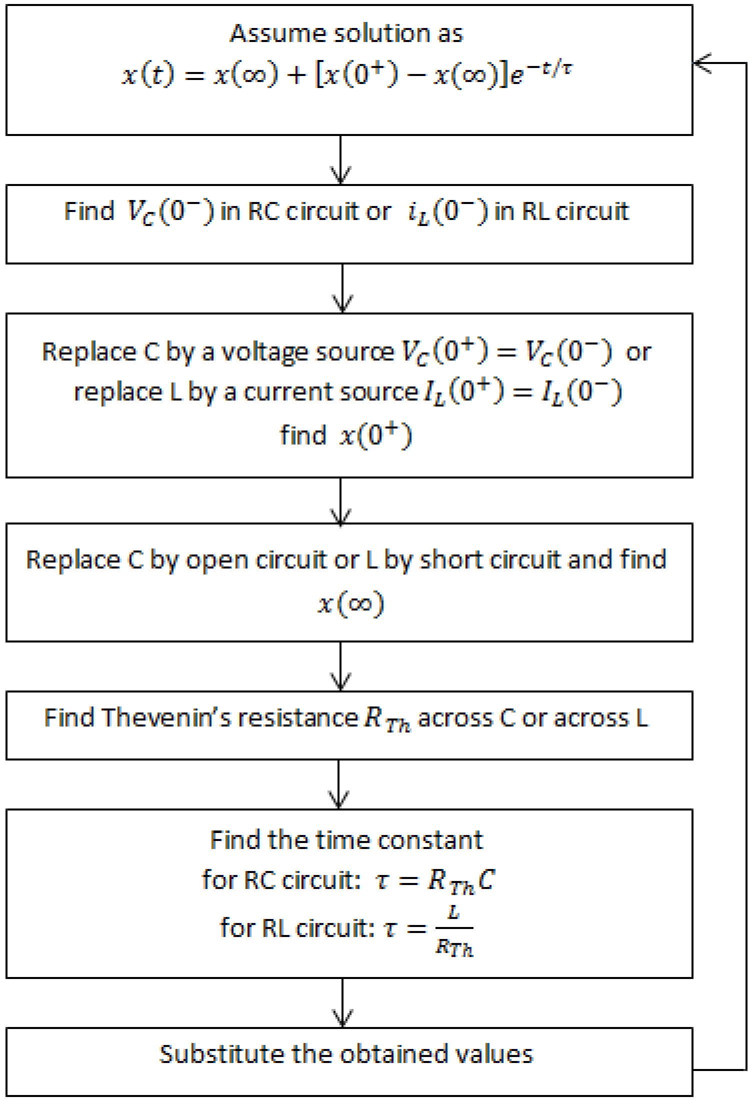


Figure 1: Step-by-Step Approach

The two approaches are illustrated by solving the current $i_0(t)$ for $t > 0$ in the circuit shown in Figure 2.

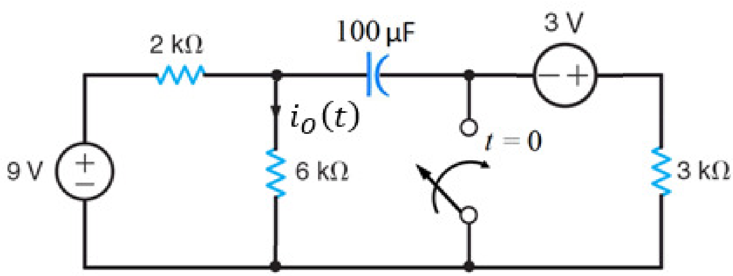
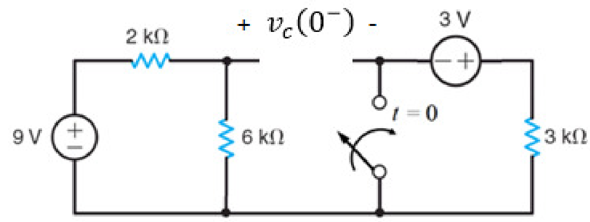


Figure 2: First-order RC circuit

1) Differential Equations Approach

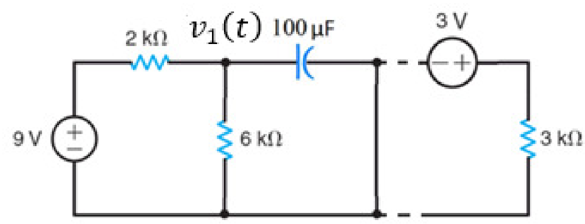
We begin by finding the voltage across the capacitor before the switching action, the capacitor acts as an open circuit as shown in the figure below.



A simple voltage division and difference in node voltages provide$v_c(0^-)$:

$$v_c(0^-) = \frac{6k}{(6+2)k}(9) - (-3) = 9.75\ V$$

After the switch closes, the circuit can be represented as shown in the figure below.



Since the requirement is to find the current through the 6 kΩ resistor, we have to find the voltage across it first. This voltage can be determined using any analysis techniques (nodal, loop, etc.). Using nodal analysis, we can write a nodal equation at node $v_1(t)$, which results in the following equation.

$$\frac{v_1(t) - 9}{2k} + \frac{v_1(t)}{6k} + C\frac{dv_1(t)}{dt} = 0$$

Simplifying, we get

$$0.5v_1(t) - 4.5 + 0.167v_1(t) + 1000C\frac{dv_1(t)}{dt} = 0$$

Collect terms to obtain

$$0.667v_1(t) + 1000C\frac{dv_1(t)}{dt} = 4.5$$

or

$$v_1(t) + 1500C\frac{dv_1(t)}{dt} = 6.75 \qquad (5)$$

The homogeneous differential equation associated with (5) is

$$v_1(t) + 1500C\frac{dv_1(t)}{dt} = 0$$

The characteristic equation is

$$1500C\lambda + 1 = 0$$

which indicates an eigenvalue of $\lambda = -1/1500C$ and characteristic mode $e^{\frac{-t}{1500C}}$. Hence, the homogeneous solution has the form

$$v_{1_h}(t) = K_1 e^{\frac{-t}{1500C}}$$

The particular solution takes the form of a constant since the forcing function is a constant:

$$v_{1_p}(t) = K_2$$

Using the form of the differential equation (5), we can obtain the value of K_2:

$$K_2 + 1500C(0) = 6.75$$

So that $K_2 = 6.75$. Since v_1 is the voltage across the capacitor, which cannot change instantaneously,

$v_1(0^+) = v_c(0^+) = v_c(0^-) = 9.75$V.

Using this initial condition along with the complete solution allows us to solve for K_1.

$$\begin{aligned} v_1(t) &= K_1 e^{\frac{-t}{1500C}} + K_2 \\ &= K_1 e^{\frac{-t}{1500C}} + 6.75 \end{aligned}$$

$$\Longrightarrow \quad 9.75 = v_1(0^+) = K_1 + 6.75$$

$$\therefore \qquad K_1 = 3$$

Hence,

$$v_1(t) = 3e^{\frac{-t}{1500C}} + 6.75$$

Finally, by Ohm's Law (and substituting $C =$1000µF) we obtain

$i_0(t) = \frac{v_1(t)}{6k} = 1.125 + 0.5e^{-6.67t}$ mA for $t > 0$

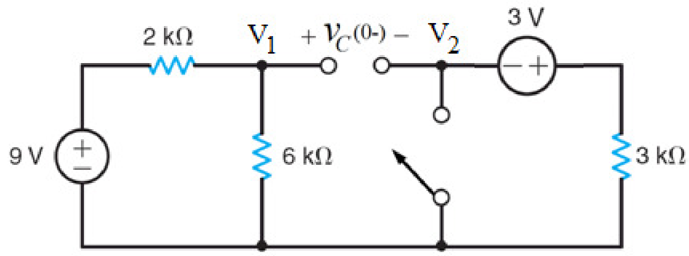
2) Step-by-Step Approach

The step-by-step approach illustrated in Figure 1 is utilized to solve the circuit for the current $i_0(t)$ that is flowing through the 6 $k\Omega$ resistor. The steps are given below.

Step 1: Assume a solution of the form

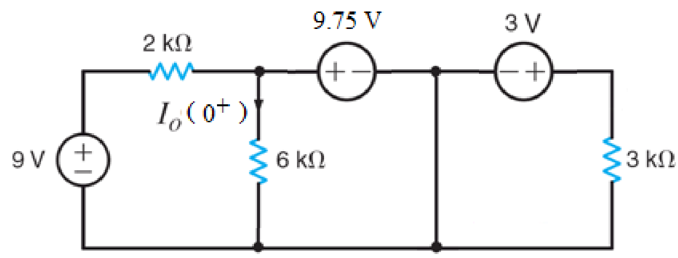
$$i_0(t) = I_0(\infty) + [I_0(0^+) - I_0(\infty)]e^{-t/\tau}$$

Step 2: Draw the circuit prior to the switching action, replace the capacitor by an open circuit, and find $V_C(0^-)$ shown below using any analysis technique. A simple voltage division and difference in node voltages provides $v_c(0^-)$



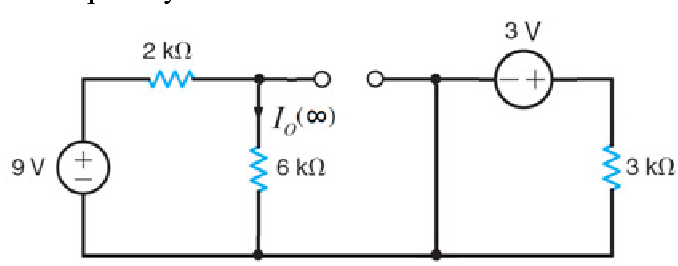
$$v_c(0^-) = \frac{6k}{(6+2)k}(9) - (-3) = 9.75 \text{ V}$$

Step 3: Draw the circuit after the switching action, replace the capacitor by a voltage source equal to $V_C(0^+) = V_C(0^-) = \ 9.75V$ and find $I_o(0^+)$ using any analysis technique. By Ohm's law, we get



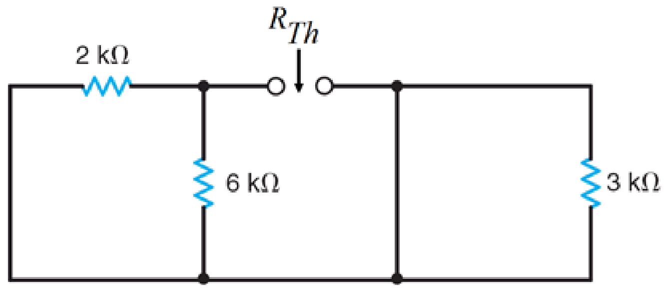
$$I_o(0^+) = \frac{9.75}{6k} = 1.625 \ mA$$

Step 4: Redraw the circuit from step 3, replace the voltage source $V_C(0^+)$ by an open circuit, and find $I_o(\infty)$ using any technique. By KVL



$$I_o(\infty) = \frac{9}{8k} = 1.125 \ mA$$

Step 5: Find Thevenin's resistance, R_{TH}, seen by the capacitor. We get



$$R_{TH} = 2\text{k} \parallel 6\text{k} = \frac{2(6)}{2+6}\text{k} = \frac{12}{8}\text{k} = 1.5\text{k}\Omega$$

Step 6: Find the time constant τ

$$\begin{aligned} \tau &= R_{TH}C \\ &= 1.5 \times 10^3 (100 \times 10^{-6}) = 0.15 \text{ s} \end{aligned}$$

Step 7: Substitute the obtained values in the assumed solution to obtain the final solution.

$$\begin{aligned} i_O(t) &= i_O(\infty) + [i_O(0^+) - i_O(\infty)]e^{-t/\tau} \\ &= 1.125 + [1.625 - 1.125]e^{-t/0.15} \\ &= 1.125 + 0.5e^{-6.67t} \text{ mA} \quad \text{for } t > 0 \end{aligned}$$

As expected, the current obtained is the same in both cases.

4. Student Performance and Assessment

The assessment of students' performance is carried out based on data obtained from two successive years. The first set of data represents the performance of students using the differential equations approach and the second set of data represents the performance of students using the step-by-step approach.

1) Differential Equations Approach

The pool included students enrolled in one section of the electric circuits course. There were 24 sophomore students in the class distributed as 3 females and 21 males. The distribution of students along with their major is shown in Table 1.

Table 1 Students distribution in the course per major

Student Major	Number of Students
Electrical Engineering	9
Computer Engineering	4
Mechanical Engineering	10
Civil Engineering	1

The average ACT score of the above students was 27. Although both electric circuits and differential equation courses are offered in the same term, all students have the required knowledge of solving first-order differential equations by the time transient analysis of first-order circuits is covered in the electric circuit class.

Students were asked to rank their knowledge in solving first-order differential equations. They ranked themselves with an average of 8 out of 10 points.

Performance Assessment

The assessment is carried out through two means: the first is through homework problems and the second through an exam. The average score on the homework was 90%, whereas in the exam was 60%. Usually students obtain higher scores in homework compared to exams as expected, and that is due to many factors, such as working on the problems with their classmates and the fact that time pressure is not present.

Misconceptions

Based on the assessment, the following are some of the mistakes that students have in solving first-order circuits using differential equations:

(a) Determination of initial conditions in a given problem.
(b) Solving for the required coefficients.
(c) Mixing of units during substitution.
(d) Feeling that the equations get too complicated.

2) Step-by-Step Approach

The pool included students enrolled in one section of the electric circuits course. There were 28 sophomore students in the class distributed as 4 females and 24 males. The distribution of students along with their major is shown in Table 2.

Table 2 Students distribution in the course per major

Student Major	Number of Students
Electrical Engineering	10
Computer Engineering	2
Mechanical Engineering	16
Civil Engineering	0

The average ACT score of the above students was 26. The case remained the same this year compared to the previous year that both electric circuits and differential equations courses are offered in the same term.

Performance Assessment

The assessment was carried out exactly the same way as the previous year through two means: the first is through homework problems and the second through an exam. The average score on homework was 95%, whereas in the exam was 80%. The fact that students obtained higher scores in homework compared to exams is expected and remained valid. That is due to many factors as mentioned earlier such as working on the problems with their classmates and time pressure is not present.

Misconceptions

Based on the assessment, the following are some of the mistakes that students have when solving first-order circuits using the general step-by-step approach:

(a) Solving for the variable to be determined at $t = 0^{-}$ instead of finding $v_c(0-)$ in RC circuits or $i_L(0-)$ in RL circuits in step 2.
(b) Not replacing the capacitor with a voltage source at $t = 0^{+}$ or not replacing the inductor with a current source at $t = 0^{+}$ required in step 3.
(c) Mixing up whether a capacitor should be replaced with a short circuit or open circuit in step 4. Likewise, with the inductor.
(d) Finding the Thevenin resistance in step 5.

5. Conclusions

A majority of students find the first-order circuit transient analysis topic to be challenging due to the interconnected devices that lead to electric circuits that must satisfy both algebraic equations as well as integral or differential relationships. The paper presents the two popular approaches for solving such circuits. The first approach involves solving the differential equations using the method of undetermined coefficients. The second approach is the general step-by-step method. Based on the assessment results, both approaches have their own advantages and disadvantages in terms of misconceptions; however, in the authors' experience, students prefer to use the step-by-step approach, although it is sometimes a longer solution compared to the differential equations approach. Students performed better using the step-by-step approach. Thus, this paper recommends the use of step-by-step approach for solving for the transient response of first-order circuits not only because students performed better using this approach, but also their attitude toward using this approach is more positive compared to its counterpart.

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