

A Circuit Approach to Teaching Skin Effect

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Abstract: This paper presents a circuits-based approach to describing skin effect. A student project in library use confirmed that the prevalent approach in electromagnetic textbooks is to introduce skin effect and skin depth as a special case of wave propagation in a good conductor. This wave approach may leave a student with the impression that skin effect is a wave phenomena. Skin effect is, however, a magnetoquasistatic effect, which can be demonstrated using only circuit analysis and Faraday's Law. A simple circuit with three equal, parallel resistors, arranged in a plane can be used to illustrate the crowding of current to the outside resistors, to the "skin". For electrical engineering students with a strong circuits analysis background, such a description is concrete, building on circuit understanding, rather than a special case of wave propagation, unrelated to the lumped circuits they have studied. This paper reviews two common routes to skin effect, the prevalent wave approach and the diffusion equation, and compares these to the simple circuit illustration. In addition, it describes the student project which confirmed the dominance of the wave approach in electromagnetic field textbooks, relating that project to an EE program outcome and the ABET 2000 Criteria 3 (g) and (i).

Introduction

The tendency of high-frequency electrical current to crowd to the surface of a conductor is well known¹. However, the basic physics behind this skin effect may be obscured by the context in which it is presented in books on electromagnetic fields. A survey performed by students as a course project, described in this paper, shows that a wave description is the most common. Not only do most textbooks present the skin effect as a special case of wave propagation in a good conductor, at least one² regards it as "inherently a wave propagation phenomenon." In contrast, this paper presents an attempt to reduce a description of the effect to its bare essentials of circuit theory and Faraday's Law of Induction, to motivate an understanding of skin effect as a quasi-static effect³ rather than a wave phenomenon.

First, the common approach to deriving skin effect and skin depth is reviewed. Then an alternative quasi-static mathematical approach is presented that yields a solution for a distributed system. Finally, analysis of a simple three-resistor circuit illustrates the basic behavior of current crowding to the outside of a conduction path in a lumped circuit. In addition, details of the student course project assignment and results are presented in the context of ABET requirements.

TEM Wave limit

A typical derivation of a uniform plane wave solution of Maxwell's Equations in a source-free region begins with the differential form of equations in phasor notation, (with $e^{j\omega t}$ suppressed, where ω is the frequency in radians/second) in the MKS system:

$$\nabla \times \hat{\mathbf{E}} = -j\omega\mu\hat{\mathbf{H}} \quad (1)$$

$$\nabla \times \hat{\mathbf{H}} = (\sigma + j\omega\epsilon)\hat{\mathbf{E}}. \quad (2)$$

Here, $\hat{\mathbf{E}}$ is the complex electric field intensity and $\hat{\mathbf{H}}$ is the complex magnetic field intensity, with the "hat" indicating phasor field vectors, following a common notation⁴. The material conductivity, permittivity, and permeability are represented by σ , ϵ , and μ , respectively.

A uniform plane wave solution to these equations, for a wave traveling in the +z direction with propagation constant γ and intrinsic wave impedance η , is

$$\hat{E}_x = E_m e^{-\gamma z} \quad (3)$$

$$\hat{H}_y = \frac{E_m}{\eta} e^{-\gamma z} \quad (4)$$

where $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$ and $\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$ and $\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$ (5)

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} \quad (6)$$

For a lossless medium, the conductivity $\sigma = 0$, and equations (3) – (6) describe a wave propagating in the +z direction without attenuation,

with a propagation constant $\gamma = j\omega\sqrt{\mu\epsilon}$ (7)

and an intrinsic wave impedance $\eta = \sqrt{\frac{\mu}{\epsilon}}$ (8)

It is often noted that the two time derivatives in Maxwell's Equations, appearing in equations (1) and (2) as the two terms with $j\omega$ factors, provide the coupling between the electric and magnetic fields that support propagating waves solutions.

Of more interest in this paper is the solution for a good conductor, defined as a material with $\sigma \gg \omega\epsilon$. Then the $j\omega\epsilon$ term in equation (2) is regarded as negligible compared to σ , and the propagation constant becomes

$$\gamma = \sqrt{j\omega\mu\sigma} = \frac{1}{\delta} + \frac{j}{\delta}, \text{ where } \delta = \text{skin depth or penetration depth} = \sqrt{\frac{2}{\omega\mu\sigma}}. \quad (9)$$

Now the solution represents a decaying wave, with decay length (over which wave amplitude is reduced by e^{-1}) of δ . The limited penetration depth, described by δ , then becomes the basis for discussion of skin effect in good conductors at high frequency.

This approach is mathematically rigorous and provides an expression for the skin depth in sinusoidal steady state. However, it may leave students with two incorrect impressions. First, it appears that the phenomenon of limited penetration of field and current into a conductor occurs in the context of a general wave solution, and thus skin effect cannot be understood using lumped circuit concepts. The second misconception is that the phenomenon exists only in sinusoidal steady state, since it was derived as a special case of a sinusoidal steady state (phasor) wave solution.

A broader approach can be developed by noting that the “good conductor” approximation neglects the displacement current density in the sinusoidal steady-state, $j\omega\epsilon\hat{\mathbf{E}}$, compared to conduction current density $\sigma\hat{\mathbf{E}}$. Thus, it must be possible to obtain an equivalent solution by neglecting displacement current initially in Maxwell’s equations, avoiding the need for a general wave solution.

Diffusion Equation

Starting from the magnetoquasistatic (MQS) form⁵ of Maxwell’s curl equations, i.e., with displacement current density neglected, in a uniform, homogenous conducting media, and substituting $\hat{\mathbf{E}} = \hat{\mathbf{J}} / \sigma$ yields

$$\nabla \times \hat{\mathbf{J}} = -\sigma\mu \frac{\partial \hat{\mathbf{H}}}{\partial t} \quad \text{and} \quad (10)$$

$$\nabla \times \hat{\mathbf{H}} = \hat{\mathbf{J}} \quad (11)$$

Consider a case in which current is flowing in the z direction, with variation of current along x, but no variation of current along y: $\hat{\mathbf{J}} = J_z(x,t)\mathbf{a}_z$, where \mathbf{a}_z indicates a unit vector in the z direction. Then equation (10) shows that the magnetic field intensity must be in the y direction: $\hat{\mathbf{H}} = H_y(x,t)\mathbf{a}_y$. Substituting these forms and combining equations (10 and (11) into one yields

$$\frac{\partial^2 J_z}{\partial x^2} = \sigma\mu \frac{\partial J_z}{\partial t} \quad (12)$$

This equation is a one-dimensional diffusion equation, with $\frac{1}{\sqrt{\sigma\mu}}$ as the diffusion

constant⁶. An initial current established on the surface of a conductor thus diffuses into a conductor, following the same process that EE students may have studied in semiconductor electronics⁷. Note that in application-oriented texts, the diffusion constant may be defined without the square root. If phasor notation is introduced for a sinusoidal steady-state solution to the diffusion equation (12), then

$$\frac{\partial^2 \hat{J}_z}{\partial x^2} = j\omega\sigma\mu\hat{J}_z \quad (13)$$

A solution has the same form as the special case of the wave equation (9):

$$\hat{J}_z(x) = \hat{J}_0 e^{-x/\delta} e^{-jx/\delta}, \quad \delta = \sqrt{\frac{2}{\omega\mu\sigma}} \quad (14)$$

This approach directly from the MQS form of Maxwell's equations provides equations in both the time domain and the frequency domain, and may relate the mathematics describing the behavior of the current to other topics studied by EE students, such as the behavior of carriers in semiconductors. However, it seems no more useful than the wave equation for providing students with an understanding of the physical basis of the phenomenon of skin effect.

Circuit Approach

A simple circuit approach can illustrate that the basic phenomenon of skin effect is the result of induced voltage from Faraday's Law. The success of the MQS approach shows that this must be true, since MQS uses static field equations except for the time variation of the magnetic field. Thus, the combination of lumped circuits and Faraday's Law of Induction must be sufficient to elucidate the essential behavior of skin effect.

Consider a three-resistor circuit with the resistors in a plane, as shown in Figure 1. The three equal, parallel resistors are fed from a current source, current $3i$. At DC, with static fields, currents, and voltages, the current splits evenly, with $i_1 = i_2 = i_3 = i$.

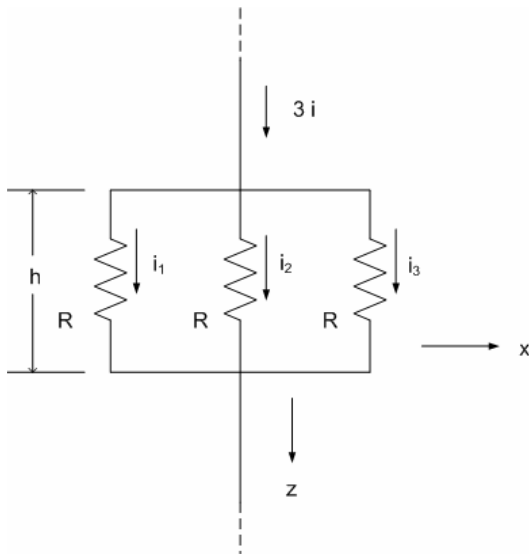


Figure 1 Three parallel resistors, top view, showing z, x directions

Figure 2 shows the same physical circuit viewed by looking down the z axis. Although the resistors and wire have cross-sectional area, they are regarded as filaments having length but no area, to keep the mathematical model simple.

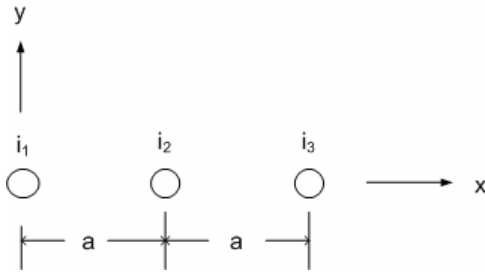


Figure 2 Three parallel resistors, end view, showing x, y directions.

If the current source varies with time, the magnetic flux through the circuit loops will vary with time, inducing a voltage around the loops. Calculating the flux at DC, then introducing a time variation of the current, yields a first order correction term to the DC solution⁸.

The paths of the magnetic fields from the three current branches are sketched in Figure 3.

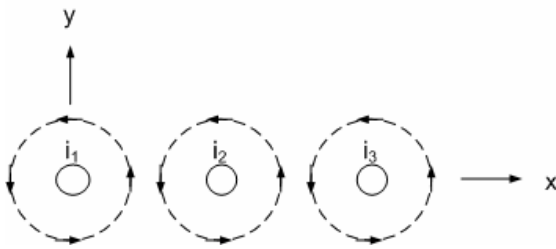


Figure 3 Three parallel resistors, end view, showing magnetic fields from each current.

The magnetic flux through the area of the x-z plane bounded by the currents i_2 and i_3 is

$$\Phi_{23} = \frac{\mu_0}{2\pi} \int_a^{2a} \int_0^h \left\{ \frac{i_1}{x} + \frac{i_2}{x-a} + \frac{i_3}{x-2a} \right\} dz dx \quad (15)$$

Here the current paths have been assumed to be filaments, with zero cross-sectional dimension, as described earlier. For the calculation in equation (15), the current i_1 is assumed to be located at $x = 0$.

At DC, with $i_1 = i_2 = i_3 = i$, the second and third terms in the integral add to zero. This cancellation can be seen from the field sketch in Figure 3 or can be derived by rearranging the integration. The resulting expression for the flux is then

$$\Phi_{23} = \frac{\mu_0 h}{2\pi} \int_a^{2a} \left\{ \frac{i}{x} \right\} dx = \frac{\mu_0 h i}{2\pi} \ln(2) \quad (16)$$

By Faraday's Law, the magnitude of the voltage induced around the loop through the center and right-hand resistor is

$$|v_{ind}(t)| = \left| \frac{d\Phi_{23}}{dt} \right| = \left| \frac{di(t)}{dt} \right| \frac{\mu_0 h}{2\pi} \ell n(2). \quad (17)$$

Now, $i(t)$ in equation (17) is regarded as a function of time, even though the expression was derived assuming DC current. Thus, this voltage must be regarded as a first order correction to the static, DC circuit solution if the current varies in time.

Consider a case of a current is ramping up in time, $i = i(t) = I_0 t/\tau$, where $I_0 = 5$ A, and $\tau = 1.0$ μ second, and the length of the circuit, $h = 1.0$ cm. Then the induced voltage is 6.9 mV. The polarity of the induced voltage could be obtained by carefully tracing the signs in Faraday's Law and the definition of voltage. However, it is easier to apply Lenz's Law, which states that an induced current (resulting from an induced voltage) always opposes the change in the flux that induced it.

Thus, if the original flux through the x-z plane is increasing in the positive y direction, from current $i(t)$ ramping up in time, then the induced voltage will produce a current that causes a magnetic flux in the negative y direction. The polarity of the voltage will be such as to cause current to circulate cw through the center and right-hand resistors seen in Figure 1. Since an identical development holds for the left-hand loop in Figure 1, the induced voltage may be represented by a voltage source in the center resistor leg, as shown in Figure 4.

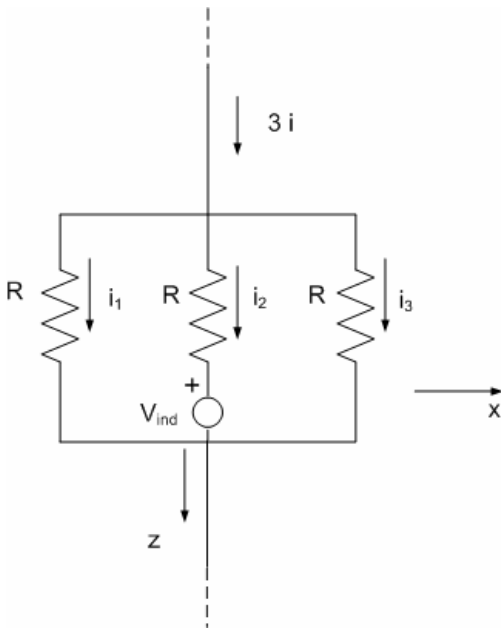


Figure 4 Three parallel resistors showing voltage source representing induced voltage.

The first correction accounting for the ramp time variation of the magnetic field is a constant volt source. But a constant voltage may be obtained from a linear current ramp through an inductor, $v_{ind} = L di/dt$. The circuit representation for the first order correction for time-varying magnetic fields can thus be represented by an inductor in the

center leg, as shown in Figure 5. For a length of the circuit $h = 1.0$ cm, the inductor $L = 1.4$ nH.

In this circuit illustration, the induced voltage from time-varying current has been shown to be equivalent, to first order, to an inductor in the center path of the current. Now EE students can readily see the implications. At DC the current division is even, with equal currents in each of the three resistors. At high frequency, more current is forced through the outer resistors compared to the equal split at DC. This unequal current division increases the overall power loss. Further, any sharp transient current, such as a step in current, will be established initially in the outer resistors, “diffusing” into the center resistor to approach an equal current split.

While this circuit approach highlights the essential physics, a lumped circuit cannot be expected to produce more than a first order correction to DC solution, compared to a distributed MQS field analysis. In the circuit, at high enough frequency, nearly all the current flows in the two outer resistors, with no further increase in power loss possible. (In the three-resistor circuit, this limit occurs at 1.5 times the dc loss.) However, in a thin piece of copper, with current crowding toward the edges, the crowding continuously increases with frequency, with no limit to the increasing power loss.

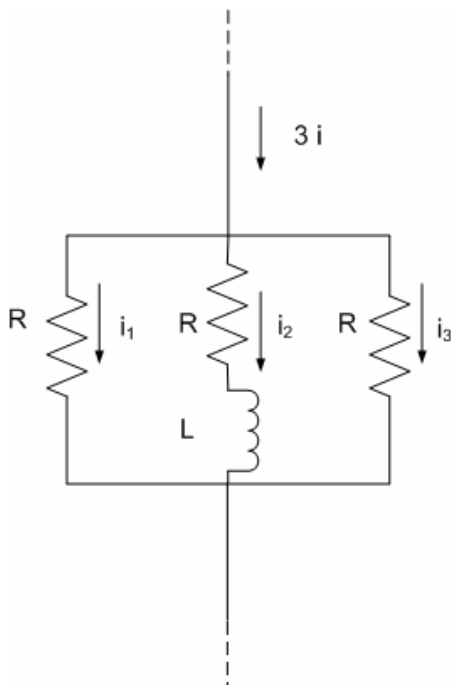


Figure 5 Three parallel resistors, with inductor representing the effect of time-varying magnetic fields from time-varying current.

In engineering literature, analysis of the effects of skin effect and other induced current effects in structures such as transformer windings is typically presented as a ratio of power loss at a frequency ω normalized to the loss at DC. This ratio can be interpreted as a ratio of AC resistance to DC resistance for the conductor⁹. The power loss for an isolated conductor carrying current in the z direction can be derived from the MQS form

of Maxwell's Equations¹⁰. The conductor is assumed to be thin enough in the y direction to neglect y-variations of current density, but show current crowding to outer edges in the x-direction, providing a distributed conductor version of the three-resistor circuit. Figure 6 shows the results of normalized power calculations for a thin copper conductor 1.0 cm wide and for the lumped circuit model of Figure 5. For the circuit, the values of the resistors were $0.14 \mu\Omega$, the low value consistent with a lumped representation of a copper sheet. The graph shows good agreement between the distributed and lumped results until about 100 radians/second. Then the curves begin to separate, with the lumped result approaching 1.5 and the distributed result continuing to rise as expected.

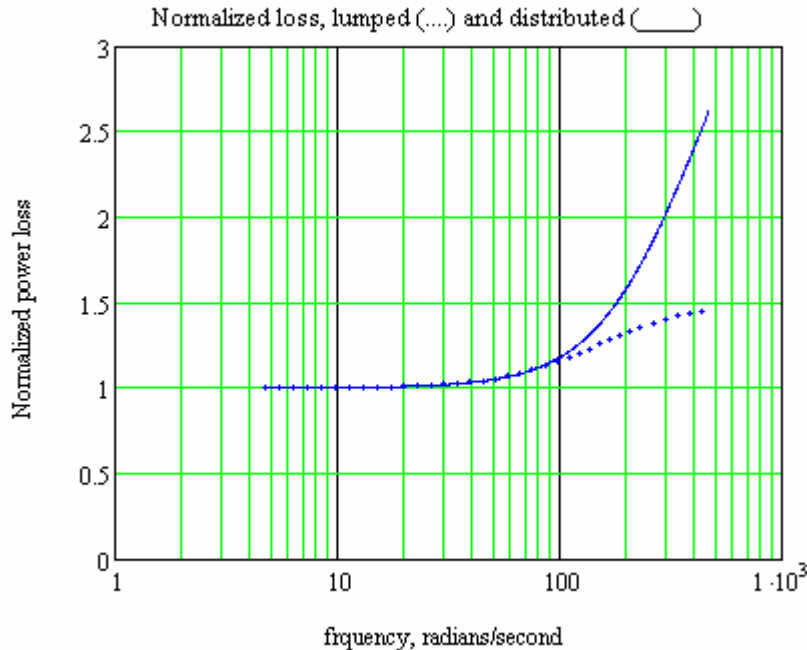


Figure 6 Comparison of power loss vs. frequency for circuit (....) and field (___) calculations.

This comparison confirms that the simple three-resistor circuit, with an inductor derived from Faraday's Law, provides a good circuit illustration of skin effect: the current shift from the inside resistor to the outside resistors increases the power loss in the circuit, just as current crowding to the skin of a conductor increases loss. Of course, it is necessary to solve a distributed fields problem, using MQS equations, to obtain expressions for current distribution and power loss that are valid over a wide frequency range. It is not necessary to derive a wave equation in order to illustrate the skin effect.

Library Project

The prevalence of the wave equation approach is revealed by information gathered by students for a book survey project in EM Fields and Waves, a Junior-level Electrical Engineering course at Indiana Tech. In this project, each student was assigned five books on electromagnetism, with publication dates spanning the decades of 1950 through 1990. Each student received an individual list of five library call numbers only. The final survey report had to include the following basic information for each book:

1. Full name of author, with affiliation.
2. Full name(s) of any co-author(s), with affiliation(s).
3. Title of book, with edition number.
4. Publisher's name with location and year of publication or copyright.
5. ISBN number (if no ISBN, then use the Library of Congress catalog number).
6. Total number of pages of text.
7. Title and number of pages of each appendix (if any).

In addition to this basic information, students were required to address questions about the topics in each book. They had to cite page numbers to support their responses to these questions.

1. What is the level of this book?
2. What system of units is used?
3. Does this book use four field vectors, **E**, **D**, **B**, and **H**, or only two of these?
4. How is the electric field intensity defined fundamentally?
5. Are the terms "electroquasistatics" and "magnetoquasistatics" used?
6. How is skin effect introduced?
7. Any general comments on organization, clarity, and figures?

This project helped students become familiar with a variety of books on electromagnetism, including some of the earlier work. For example, a Dover reprint of Heaviside's works was included in the project. In particular, the project provided an opportunity for students to search several books for information on topics, and to summarize their findings, as they might be required to do in engineering practice. This activity was in contrast to using the one course text to find help with an end-of-chapter exercise, as they do in the classroom. In this manner, this project supported ABET criteria 3g, addressing communication, and 3i, addressing life-long learning¹¹. In the context of assessment, this project supported Indiana Tech's program outcome 4.1, that "Seniors be able to critically read engineering literature."¹² Most students seemed to enjoy the project, with its bit of treasure hunt aspect, starting with only the call number. The student whose list included the Heaviside work was especially excited about his library "find".

Of particular interest here is the information obtained in response to question #6. The project covered 30 books. Of these, 22 contain at least a qualitative description of the skin effect. The limiting case of wave propagation in a good conductor is used in 15 of these to describe skin effect and skin depth.

Conclusion

A survey of electromagnetic books revealed that approximately 2/3 of them address skin effect as a special case of wave propagation. This approach, while rigorous, may be misleading because it obscures the magnetoquasistatic (MQS) approximation made in the special case of wave propagation in a good conductor. It leaves the impression that skin effect is a wave phenomenon, occurring at sinusoidal steady state. At the other extreme, a simple lumped circuit analysis, combined with Faraday's Law, can be used to illustrate

the basic phenomenon of current crowding to the outside of a conductor. While the circuit approach illustrates the essential physics of the skin effect, the diffusion equation, obtained directly from Maxwell's Equations in MQS form, is necessary to obtain valid solutions for distributed problems. Ideally, students should be exposed to this range of development for skin effect – circuits through diffusion to wave propagation – to reinforce the sense of unity and relevance of lumped and distributed analysis.

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Biography

JAMES H. SPREEN received his Bachelor's degree in Engineering from Case Institute of Technology and his PhD in Electrical Engineering from MIT. He has held engineering positions at several industries, including IBM. Dr. Spreen, a member of ASEE, ECS, and a Senior Member of the IEEE, is presently an Associate Professor of Electrical Engineering at Indiana Institute of Technology.